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Effect of Microwave Radiation on the Ionized Gas behind a Strong Normal Shock Wave

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An analytical study of the interaction between microwaves and a fully ionized gas behind a normal shock wave is presented The governing differential equations derived by a quasi-steady analysis are used to obtain the nondimensional parameters and integrated to yield the "first integrals" of the momentum and energy equations A modified Prandtl relation is derived from which the final equilibrium state of the ionized gas can be found Ionization is seen to increase the final equilibrium density, whereas microwave heating tends to reduce its value It is further shown that thermal choking of the ionized gas occurs at a critical microwave power level, which is a function of the Mach number ahead of the shock wave, the reflection coefficient of the microwave, and the ionization potential expressed in terms of the neutral gas temperature ahead of the shock wave At higher power levels, the shock wave moves upstream allowing additional microwave heating of the ionized gas A formula is given for the determination of this equivalent higher Mach number At a sufficiently high power level (but slightly below the critical value), the mean gas temperature is shown to attain a peak value higher than its final equilibrium value This peak temperature actually represents the upper bound of the mean gas temperature for a given initial Mach number

I Introduction

IN flowing through a strong normal shock wave, a neutral gas can be ionized When microwaves are directed at the ionized gas along the flow direction, interaction of the ionized gas with the microwaves will occur The present study of the interaction problem will be concerned with the determination of the thermodynamic state (pressure, density, and temperature) the flow state (velocity) of the gas, and the reflection and propagation of the microwaves In view of the well-known thermal choking phenomenon, it is of interest to determine the microwave power level at which the choking will occur and to see what happens to the flow at a higher-power level

Evidently, this mathematical problem is a very difficult one Even the simpler problem of microwave propagation in a medium with fixed nonuniform electron distribution is known to be a formidable one and few exact solutions are available Therefore, in the present study, only the case where complete, or nearly complete ionization is achieved by the shock wave is considered By using the conservation laws and certain integral relations, it is shown that the final equilibrium state of the gas can be determined without difficulty In addition, some physical features of the interaction problem are discussed in some detail

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II Basic Equations—Nondimensional Parameters

The incident microwave field is assumed to be plane polarized (in x-y plane) and is of the form $E_1 \exp\{i(\omega t - kx)\}$, where E_1 is a real constant, and no steady electric or magnetic field is present Propagation of microwaves into a nonuniform plasma generally gives rise to reflected waves ahead of the shock wave of the form $E_1\tilde{R} \exp\{i(\omega t + kx)\}$, where $\tilde{R} = R \exp(i\theta)$, R being the reflection coefficient and θ the phase shift angle The transmitted waves $E(x) \exp(i\omega t)$ will be gradually damped out by the finite resistance of the ionized gas, and a portion of the incident microwave energy will appear as thermal energy and kinetic energy of the gas These are the main features of the interaction between the microwaves and the ionized gas Others such as the possible excitation of the longitudinal electrostatic waves and higher harmonics of the transverse electromagnetic waves will not be considered

To treat the interaction problem, the microwaves and the flow of the ionized gas must be considered as one "system" In a quasi steady state, the physical quantities of the system may be conveniently separated into time-averaged (or mean) values and periodic fluctuations [1] The mean physical quantities can be shown to be governed by the "flow equations" obtained by time-averaging the complete nonsteady macroscopic equations By subtracting these flow equations from the nonsteady macroscopic equations, the resultant time-dependent equations, called the "field equations," govern the oscillations of the system These two sets of equations together with Maxwell equations constitute a determinate system of equations for the interaction problem In order to render the analysis tractable, however, simplifications of these equations can be made by adopting proper assumptions, as carried out in Ref 1

Let the mass density, the number density, the temperature, and the pressure of the neutral gas ahead of the shock wave be denoted by ρ_1, n_1, T_1, and p_1, respectively The flow direction is taken to be along the positive x axis, and the velocity of the neutral gas is u_1, a constant Behind the shock wave of proper strength, ionization of the neutral gas is taken to be complete, or nearly complete For simplicity in the analysis only monatomic neutral gases are considered, but extension to other neutral gases presents no essential difficulty

Behind the shock wave, the mean flow quantities (hereafter called flow quantities for brevity) are denoted by n_i, T_i, p_i, u_i, and n , T_e, p , u for ions and electrons, respectively Charge separation effects will not be considered Thus, $n_i = n = n$, and $u_i = u = u$ Because of the vast difference in mass between ions and electrons, the ion velocity is essentially the flow velocity u By the same reasoning, the velocity fluctuations v' are due to the transverse oscillations of the electrons These oscillations yield the current $\bar{\jmath}$, which satisfied the current equation

$$\frac{m_e c^2}{n\, e^2}\frac{\partial \bar{\jmath}}{\partial t} = \bar{E} - \eta \bar{\jmath} \tag{1}$$

where η is the electrical resistivity of the ionized gas, and m_e is the electron mass (ion mass is denoted by m_i)

For the present problem, the flow equations are the continuity equation, the momentum equation, the energy equation, and the equation for energy exchange between electrons and ions as given below:

$$(d/dx)(nu) = 0 \tag{2}$$

$$(d/dx)(m_i n u^2 + nkT_i + nkT_e) = <\bar{\jmath} \times \bar{B}> \tag{3}$$

$$\frac{d}{dx}\left\{\frac{1}{2}nu(m_i u^2 + m_e < v'^2 >) + \frac{\gamma}{\gamma - 1} nuk(T_i + T\,)\right\} = <\bar{\jmath}\ \bar{E}> \tag{4}$$

$$\frac{d}{dx}\left(\frac{1}{2}m_i n u^2 + \frac{\gamma}{\gamma - 1} nukT_i\right) = \frac{6m_e}{m_i}\nu nk(T\ - T_i) \tag{5}$$

where the symbol $\langle\rangle$ denotes the time average, and ν is the collision frequency of electrons and is related to the resistivity η by the relation $\eta = m\, c^2\nu/n\, e^2$ [2] These equations have been derived rigorously in Ref 1 The equation of state is $p = p_i + p_e = kn(T + T_i)$ All electromagnetic quantities are in electromagnetic units (the electron charge is $-e/c$) The energy transfer rate from electrons to ions as given in Eq (5) is obtained from Ref 3

In the above equations, the time averages can be expressed in the following form by the known technique[4]:

$$\langle\bar{\jmath} \times \bar{B}\rangle = -\frac{1}{16\pi}\frac{\omega_p^2}{\nu^2 + \omega^2}\frac{dE^2}{dx} + O\left(\frac{\nu}{\omega}\right) \tag{6}$$

$$m\, n\langle v'^2\rangle = \frac{1}{8\pi}\frac{\omega_p^2}{\nu^2 + \omega^2}E^2 \tag{7}$$

$$\langle\bar{\jmath}\ \bar{E}\rangle = \frac{1}{8\pi}\frac{\nu\omega_p^2}{\nu^2 + \omega^2}E^2 \tag{8}$$

where $E^2 = E_r^2 + E_i^2$ (E_r and E_i are the real and imaginary parts of E, and E is now in electrostatic units), and ω_p is the plasma frequency defined by

$$\omega_p^2 = 4\pi n_e e^2/m_e \tag{9}$$

In Eq (6), the terms denoted by $O(\nu/\omega)$ are higher order terms in (ν/ω) and can be omitted for $(\nu/\omega) \ll 1$

From the Maxwell equations (in electromagnetic units)

$$-(\partial\bar{B}/\partial t) = \nabla \times \bar{E} \tag{10}$$

$$4\pi\bar{\jmath} + (1/c)^2(\partial\bar{E}/\partial t) = \nabla \times \bar{B} \tag{11}$$

and the current Eq (1), the following equation can be obtained:

$$\frac{d^2E}{dx^2} + \left(\frac{\omega}{c}\right)^2\left\{1 - \frac{\omega_p^2(i\nu + \omega)}{(\nu^2 + \omega^2)\omega}\right\}E = 0 \tag{12}$$

This equation governs the propagation of the microwaves

Eq (2) shows that $nu = n_1u_1$, a constant Equations (3–5 and 12) form a determinate system for the four unknowns u, T_i, T , and E The complex reflection R is also unknown and may be regarded as an eigen value of the problem

From the governing differential equations, nondimensional parameters pertinent to the problem can be obtained n_1, u_1, T_1, and E_1 are used as the typical quantities, and for the typical length $l_1 = c/\omega$ is used The following nondimensional parameters are obtained:

Power (density) Ratio

$$\sigma = \frac{E_1^2c}{8\pi\rho_1u_1^3} = \frac{E_1^2c}{8\pi m_i n_1 u_1^3} \tag{13}$$

Mach Number

$$M_1 = u_1/a_1 \tag{14}$$

Frequency Ratio

$$\Omega_p = \omega_{p0}/\omega \tag{15}$$

Collision Frequency Parameter

$$\Omega_c = \nu_0/\omega \tag{16}$$

Energy Transfer Parameter

$$\Omega_{ei} = m\ c\nu_0/m_i\omega u_1 \tag{17}$$

where $a_1^2 = 2\gamma kT_1/m_i$, and the subscripts 0 refer to the conditions of the ionized gas immediately behind the shock The Mach number defined differs from the conventional one for neutral gas by a factor of $2^{-1/2}$ The energy exchange parameter pertains to the energy transfer from electrons to ions by elastic collisions Let T_{ion} be the ionization potential of the gas in °K [5] Then an additional parameter is $\tau_{\text{ion}} = T_{\text{ion}}/T_1$

The nondimensional variables are defined by

$$\mu = \frac{u}{u_1} \qquad \tau_e = \frac{T_e}{T_1} \qquad \tau_i = \frac{T_i}{T_1}$$
$$\epsilon = \frac{E}{E_1} \qquad \xi = \frac{x}{l_1} = \frac{x\omega}{c} \tag{18}$$

Now the assumption is made that the microwave interaction can be ignored across the shock front The state of the ionized gas immediately behind the shock front can be specified by the velocity u_0, and the mean temperature T_0 of the gas defined by $T_0 = \frac{1}{2}(T_{i0} + T\ _0)$ By integrating Eqs (3) and (4) across the shock front, the following two equations for u_0 and T_0, or μ_0 and τ_0 in nondimensional form, are obtained (γ being taken to be $\frac{5}{3}$):

$$\mu_0 + \frac{3}{5M_1^2}\frac{\tau_0}{\mu_0} = 1 + \frac{3}{10M_1^2} \tag{19}$$

$$\frac{1}{2}\mu_0^2 + \frac{3}{2M_1^2}\tau_0 = \frac{1}{2} + \frac{3}{4M_1^2} - \frac{3}{10M_1^2}\tau_{i\,n} \quad (20)$$

The last term in Eq (20) accounts for the ionization energy kT_{ion} required to ionize the gas completely Solving Eqs (19) and (20) for μ_0 and τ_0

$$\mu_0 = \frac{5}{8}\left(1 + \frac{3}{10M_1^2}\right) - \frac{3}{8}\left\{\left(1 - \frac{1}{2M_1^2}\right)^2 + \frac{16\tau_{ion}}{15M_1^2}\right\}^{1/2} \quad (21)$$

$$\tau_0 = \frac{5M_1^2}{3}\mu_0\left(1 + \frac{3}{10M_1^2} - \mu_0\right) \quad (22)$$

If the temperatures of the electrons and the ions immediately behind the shock front are assumed to be the same, the values of μ_0 and τ_0 given by Eqs (21) and (22) can be taken as the initial values of u, τ_i, and τ_e in the interaction problem Otherwise, μ_0 and τ_0 may be regarded as the reference velocity and temperature for the ionized gas

The collision frequency for a fully ionized gas (with ionic charge $Z = 1$) is[2]

$$\nu = 1\ 85 n \ln\Lambda/(\tau\, T_1)^{3/2} \quad (23)$$

It is found from Eq (23) and the continuity equation that

$$\frac{\omega_p^2}{\nu^2 + \omega^2} = \Omega_p^2\frac{\mu_0}{\mu\{1 + \Omega_c^2(\nu/\nu_0)^2\}} \quad (24)$$

$$\frac{\nu}{\nu_0} = \frac{\mu_0}{\mu}\left(\frac{\tau_0}{\tau}\right)^{3/2} \quad (25)$$

$$\mu_0\Omega_i = \frac{m_e c\nu_0}{m_i\omega u_1} = \left(\frac{m_e}{m_i}\right)^{1/2}\frac{c\nu_0}{\omega M_0}\left(\frac{m_e}{2\gamma k T_0}\right)^{1/2} \quad (26)$$

where $M_0 = u_0/(2\gamma k\tau_0 T_1)^{1/2}$ Equation (26) indicates that the physically meaningful typical length for energy transfer between electrons and ions is essentially the mean free path of the electrons multiplied by the ratio $(m_i/m\ )^{1/2}$ It is noted that for given Ω_p, $\Omega\ $, σ, Ω_i, μ_0, and τ_0, all the quantities n_1, u_1, T_1, M_1, n_0, u_0, T_0, T_{ion}, and E_1^2c can be calculated for chosen ω and m_i Thus, the interaction problem can be specified by a proper choice of these six parameters

By using the foregoing relations and the assumption that $\Omega_c \ll 1$, the governing differential equations, (3–5 and 12), can be written in the form as follows:

$$\frac{d}{d\xi}\left(\mu + \frac{3}{10M_1^2}\frac{\tau_i + \tau_e}{\mu}\right) = -\frac{1}{2}\mu_0\Omega_p^2\sigma\frac{u_1}{c\mu}\frac{d}{d\xi}(\epsilon^2) \quad (27)$$

$$\frac{d}{d\xi}\left\{\frac{1}{2}\mu^2 + \frac{3}{4M_1^2}(\tau_i + \tau\ ) + \frac{1}{2}\mu_0\Omega_p^2\sigma\frac{u_1}{c}\frac{\epsilon^2}{\mu}\right\} = \mu_0^2\tau_0^{3/2}\Omega\ \Omega_p^2\sigma\frac{\epsilon^2}{\mu^2\tau_e^{3/2}} \quad (28)$$

$$\frac{d}{d\xi}\left(\frac{1}{2}\mu^2 + \frac{3}{4M_1^2}\tau_i\right) = \frac{18}{5M_1^2}\mu_0\tau_0^{3/2}\Omega_i\frac{\tau_e - \tau_i}{\mu^2\tau^{\ 3/2}} \quad (29)$$

$$\frac{d^2\epsilon_r}{d\xi^2} + \epsilon - \frac{\mu_0\Omega_p^2}{\mu}\left\{\epsilon_r - \mu_0\tau_0^{3/2}\Omega_c\left(\frac{\epsilon_i}{\mu\tau_e^{3/2}}\right)\right\} = 0 \quad (30)$$

$$\frac{d^2\epsilon_i}{d\xi^2} + \epsilon_i - \frac{\mu_0\Omega_p^2}{\mu}\left\{\epsilon_i + \mu_0\tau_0^{3/2}\Omega_e\left(\frac{\epsilon_r}{\mu\tau^{\ 3/2}}\right)\right\} = 0 \quad (31)$$

To solve this problem as an initial value problem, a choice of the reflection coefficient R and the phase shift θ of the microwaves must be made, since

$$\epsilon_{r0} = 1 + R\cos\theta \qquad \epsilon_{i0} = R\sin\theta \quad (32)$$

$$\left(\frac{d\epsilon_r}{d\xi}\right)_0 = -R\sin\theta \qquad \left(\frac{d\epsilon_i}{d\xi}\right)_0 = -1 + R\cos\theta \quad (33)$$

These chosen values of R and θ may have to be adjusted until they become compatible with the final state of gas (see Sec IV) and the electron distribution $n(\xi)$ Thus, an iterative procedure is necessary, and this is evidently difficult to carry out Fortunately, for the present problem the reflection coefficient R can be obtained approximately by observing that most of the reflection of the microwaves occurs at the steep front of electron distribution Thus, only minor adjustments are needed In this manner, some approximate solutions of the interaction problem have been obtained by using analog computers[6]

III "First Integrals" of the Momentum and Energy Equations

Because of the nonlinear nature of the problem, analytical solutions appear to be out of the question However, some physical features of the problem can be understood without these detailed solutions This can be achieved conveniently by using the "first integrals" of the momentum and energy equations derived below

From the Maxwell equations, (10) and (11), it is found that

$$4\pi c^2\bar{\jmath} \times \bar{B} = -\bar{E}\times\nabla\times\bar{E} - c^2\bar{B}\times\nabla\times\bar{B} - (\partial/\partial t)(\bar{E}\times\bar{B}) \quad (34)$$

$$= -\tfrac{1}{2}\,\mathrm{grad}(E^2 + c^2B^2) - (\partial/\partial t)(\bar{E}\times\bar{B})$$

since only transverse field components depending on x are present

Time averaging of the foregoing equation yields for the quasi-steady case

$$\langle\bar{\jmath}\times\bar{B}\rangle = -\frac{1}{16\pi c^2}\,\mathrm{grad}(E^2 + c^2B^2) = -\frac{1}{16\pi c^2}\,\mathrm{grad}\left\{E^2 + \left(\frac{c}{\omega}\right)^2\frac{dE}{dx}\frac{dE^*}{dx}\right\} \quad (35)$$

where E^* is the complex conjugate to E

Similarly, it is found that

$$\langle\bar{E}\ \bar{\jmath}\rangle = \frac{1}{8\pi\omega}\,\mathrm{div}\left(E_r\frac{dE_i}{dx} - E_i\frac{dE_r}{dx}\right) \quad (36)$$

The first integrals of the momentum and energy equations can be easily obtained by using Eqs (35) and (36) These equations are convenient for analysis of the final state of the gas and the phenomenon of thermal choking For this analysis, it is sufficient to take the electron temperature to be the same as the ion temperature This assumption for the electron and ion temperatures may also be considered as a first approximation to the actual problem in which the electron temperature is generally higher than the ion temperature

Eqs (3) and (4), with $T_i = T$ and $\gamma = \frac{5}{3}$, become, by using Eqs (35) and (36),

$$\frac{d}{d\xi}\left\{\mu + \frac{3}{5M_1^2}\frac{\tau}{\mu} + \frac{1}{2}\left(\frac{\mu_1}{c}\right)\sigma\left(\epsilon^2 + \frac{d\epsilon}{d\xi}\frac{d\epsilon^*}{d\xi}\right)\right\} = 0 \quad (37)$$

$$\frac{d}{d\xi}\left\{\frac{1}{2}\mu^2 + \frac{3}{2M_1^2}\tau + \sigma\left(\epsilon_i\frac{d\epsilon_r}{d\xi} - \epsilon_r\frac{d\epsilon_i}{d\xi}\right) + \frac{1}{2}\left(\frac{u_1}{c}\right)\sigma\frac{\mu\omega_p^2}{\nu^2 + \omega^2}\epsilon^2\right\} = 0 \quad (38)$$

Integration of the above equations gives

$$\mu + \frac{3}{5M_1^2}\frac{\tau}{\mu} + \frac{1}{2}\left(\frac{u_1}{c}\right)\sigma\left(\epsilon^2 + \frac{d\epsilon}{d\xi}\frac{d\epsilon^*}{d\xi}\right) = 1 + \frac{3}{10M_1^2} + (1 + R^2)\left(\frac{u_1}{c}\right)\sigma \quad (39)$$

$$\frac{1}{2}\mu^2 + \frac{3}{2M_1^2}\tau + \sigma\left(\epsilon_i \frac{d\epsilon_r}{d\xi} - \epsilon \frac{d\epsilon_i}{d\xi}\right) + \frac{1}{2}\left(\frac{u_1}{c}\right)\sigma \frac{\mu\omega_p^2}{\nu^2+\omega^2}\epsilon^2 = \frac{1}{2} + \frac{3}{4M_1^2} + (1-R^2)\sigma - \frac{3}{10M_1^2}\tau_{\text{ion}} \quad (40)$$

These two equations are called the first integrals of the momentum and energy equations, respectively The right-hand side of Eq (39) is the total momentum passing into the ionized gas far downstream, whereas the right-hand side of Eq (40) represents the total energy It is of interest to note that the reflection of the microwaves reduces the amount of energy absorption rate by R^2, but it increases the microwave radiation pressure of the gas by the ratio $1 + R^2$

When the power ratio σ is of the order 1, all terms in Eqs (39) and (40) with the factor $u_1\sigma/c$ may be omitted because $u_1 \ll c$ Thus,

$$\mu + \frac{3\tau}{5M_1^2\mu} = 1 + \frac{3}{10M_1^2} \quad (41)$$

$$\frac{1}{2}\mu^2 + \frac{3}{2M_1^2}\tau + \sigma\left(\epsilon_i \frac{d\epsilon_r}{d\xi} - \epsilon_r \frac{d\epsilon_i}{d\xi}\right) = \frac{1}{2} + \frac{3}{4M_1^2} + (1-R^2)\sigma - \frac{3}{10M_1^2}\tau_{\text{ion}} \quad (42)$$

IV Some Features of the Interaction Phenomenon

In this section some general features of the problem are discussed by using the first integrals of the momentum and energy equations A modified Prandtl relation useful for determining the final equilibrium thermodynamic and flow state of the gas will be derived first Then the phenomenon of thermal choking, the "moving shock wave" solution, and the "peak temperature" of the gas will be discussed

Final Equilibrium State of the Gas; Modified Prandtl Relation

Let a critical speed of a sound a_1^* and a critical Mach number M_1^* be defined by

$$\frac{1}{2} + \frac{3}{4M_1^2} = \frac{2a_1^{*2}}{u_1^2} = \frac{2}{M_1^{*2}} \quad (43)$$

The final state of the gas is specified by μ_2 and τ_2, which satisfy, from Eqs (41) and (42),

$$\mu_2 + \frac{3}{5M_1^2}\frac{\tau_2}{\mu_2} = 1 + \frac{3}{10M_1^2} \quad (44)$$

$$\frac{1}{2}\mu_2^2 + \frac{3}{2M_1^2}\tau_2 = \frac{1}{2} + \frac{3}{4M_1^2} + (1-R^2)\sigma - \frac{3\tau_{\text{ion}}}{10M_1^2} \quad (45)$$

From the foregoing three equations, it is found that μ_2 satisfies

$$(\mu_2 - 1)\left(\mu_2 - \frac{1}{M_1^{*2}}\right) = \frac{\tau_{\text{ion}}}{10}\left(\frac{4}{M_1^{*2}} - 1\right) - \frac{1}{2}(1-R^2)\sigma \quad (46)$$

This is the modified Prandtl relation When $\tau_{\text{ion}} = \sigma = 0$, it reduces to the Prandtl relation for normal shock wave [7] Solving for μ_2 from Eq (46),

$$2\mu_2 = \left(1 + \frac{1}{M_1^{*2}}\right) \pm \left\{\left(1 - \frac{1}{M_1^{*2}}\right)^2 - 2(1-R^2)\sigma + \frac{2\tau_{\text{ion}}}{5}\left(\frac{4}{M_1^{*2}} - 1\right)\right\}^{1/2} \quad (47)$$

or

$$2\mu_2 = \frac{5}{8}\left(1 + \frac{3}{10M_1^2}\right) \pm \frac{3}{8}\left\{\left(1 - \frac{1}{2M_1^2}\right)^2 - 2(1-R^2)\sigma + \frac{16\tau_{\text{ion}}}{15M_1^2}\right\}^{1/2} \quad (48)$$

For a normal shock wave without ionization and heat addition, μ_2 is known to be equal to M_1^{*-2}, i e , the minus sign in Eq (47) should be used Thus, ionization tends to reduce the final velocity, whereas microwave heat addition acts to increase its value

Thermal Choking; Moving Shock Wave Solution

Equation (47) shows that an upper limit of the heat absorption parameter $(1 - R^2)\sigma$ exists above which μ_2 becomes complex and no physical solution can be obtained This occurs when

$$(1-R^2)\sigma = \frac{1}{2}\left\{\left(1 - \frac{1}{M_1^{*2}}\right)^2 + \frac{2\tau_{\text{ion}}}{5}\left(\frac{4}{M_1^{*2}} - 1\right)\right\}^{1/2} \quad (49)$$

and

$$\mu_2 = \frac{1}{2}\left(1 + \frac{1}{M_1^{*2}}\right) \quad (50)$$

Since the final Mach number M_2 is given by

$$M_2^2 = \left(\frac{u_2}{a_2}\right)^2 = \frac{(\mu_2 M_1)^2}{\tau_2} = \frac{1}{1 + \frac{4}{3}\left\{[1 + (1/M_1^{*2})](1/\mu_2) - 2\right\}} \quad (51)$$

the limit (or critical value) of the heat absorption will be reached when M_2 is equal to 1 This phenomenon is known as thermal choking [8] Equation (49) can be written as

$$(1 - R_c^{\,2})\sigma_{cr} = \frac{9}{32}\left(1 - \frac{1}{2M_1^2}\right)^2 + \frac{3\tau_i}{10M_1^2} \quad (52)$$

A question may be raised: What happens if σ is increased to a value beyond the critical value σ ? A possible solution is obtained by assuming the shock wave to move upstream at a constant speed and the flow far downstream to remain choked It is evident that, mathematically, by a coordinate transformation the shock wave can be made stationary in a new coordinate system Thus, this moving-shock wave solution is equivalent to another stationary shock wave solution at a higher Mach number To determine this new Mach number, let a new power ratio σ' independent of the coordinate transformation be defined by

$$\sigma' = \frac{m_i^{1/2}E_1^2 c}{8\pi n_1(2\gamma k T_1)^{3/2}} \quad (53)$$

Then, $\sigma = \sigma' M_1^{-3}$, and Eq (49) may be written as

$$\frac{(1-R^2)\sigma'}{M_1^3} = \frac{9}{32}\left(1 - \frac{1}{2M_1^2}\right)^2 + \frac{3\tau_{\text{ion}}}{10M_1^2} \quad (54)$$

Let σ_{cr}' be the critical power ratio for Mach number M_1, and for a value of $\sigma'(\sigma' > \sigma_{cr})$ the equivalent Mach number be XM_1 Then X satisfies the following equation:

$$\frac{(1-R^2)\sigma'}{X^3M_1^3} = \frac{9}{32}\left(1 - \frac{1}{2X^2M_1^2}\right)^2 + \frac{3\tau_{\text{ion}}}{X^2M_1^2} \quad (55)$$

This equation may be used to obtain X for given σ', R, M_1, and τ_{ion} However, R cannot be chosen arbitrarily as mentioned earlier Hence, an iteration procedure must be used in using Eq (55) to solve for X

Equation (55) can be used to determine the amount of power (density) absorption $(1 - R^2)\sigma'$ for given M_1, τ_{ion}, and X For example, let $\tau_{\text{ion}} = 124\ 3$, $M_1 = 10$ and $X = 1$ Then

Eq (55) gives $(1 - R^2)\sigma'/M_1^3 = 0\ 551$ Now if X is taken to be 1 5, then $(1 - R^2)\sigma'/M_1^3 = 1\ 05$ This means that, if the shock moves upstream at 50% of the velocity of the gas ahead of the shock wave, almost twice as much power absorption can be achieved by the gas downstream

Peak Temperature of the Gas

From Eqs (27) and (28), solving for $d\mu/d\xi$ and $d\tau/d\xi$ (omitting terms with the factor $u_1\sigma/c$),

$$\frac{d\mu}{d\xi} = \frac{4M_1^2\mu}{3(\tau - M_1^2\mu^2)} f(\epsilon,\mu) \tag{56}$$

$$\frac{d\tau}{d\xi} = \frac{2(3\tau - 5M_1^2\mu^2)M_1^2}{9(\tau - M_1^2\mu^2)} f(\epsilon,\mu) \tag{57}$$

where τ is the mean temperature of the gas,

$$\tau = \tfrac{1}{2}(\tau + \tau_i)$$

$$f(\epsilon,\mu) = \mu_0^2\tau_0^{3/2}\Omega_c\Omega_p^2\sigma \frac{\epsilon^2}{\mu^2\tau_e^{3/2}}$$

It is seen from Eq (57) that $(d\tau/d\xi) = 0$ when $3\tau - 5M_1^2\mu^2 = 0$ This means a peak "mean temperature" may exist in the flow with a value higher than the final equilibrium tem perature By using Eq (41) and the relation $3\tau - 5M_1^2\mu^2 = 0$ the values of μ and τ are

$$\mu_p = \frac{1}{2}\left(1 + \frac{3}{10M_1^2}\right) \tag{58}$$

$$\tau_p = \frac{5}{12} M_1 \left(1 + \frac{3}{10M_1^2}\right)^2 \tag{59}$$

From Eq (45), it is found that when

$$(1 - R^2)\sigma = \frac{1}{4} + \frac{3}{10M_1^2}(\tau_{\text{ion}} - 1) + \frac{27}{400M_1^4} \tag{60}$$

this peak mean temperature coincides with the final equilibrium temperature It follows that, for $(1 - R^2)\ \sigma$ smaller than that given by Eq (60), the mean gas temperature increases monatomically downstream and approaches the final equilibrium temperature asymptotically On the other hand, for higher $(1 - R^2)\sigma$, the mean gas temperature monatomically increases first until it reaches the peak temperature given by Eq (59) Beyond this point, the mean temperature will drop from τ_p to its final equilibrium value far downstream This is because in Eq (56), $3\tau - 5M_1^2\mu^2$ changes its sign, while μ is always a monatomically increasing function

This peak temperature, which is a function of the initial Mach number M_1 only, represents the maximum possible mean temperature attainable by the gas This can be seen from Eq (41), which may be written in the following form:

$$\frac{3}{5M_1^2}\tau_p = \mu_p\left\{\left(1 + \frac{3}{10M_1^2}\right) - \mu_p\right\}$$

The maximum of $3\tau_p/5M_1^2$ is found to occur when τ_p has the value given by Eq (58)

The significance of the foregoing result may be explained as follows: if the gas remains to be an ideal one behind the shock wave, its temperature would be

$$\tau_i = \frac{5}{8} M_1^2 \left(1 + \frac{3}{2M_1^2}\right)\left(1 - \frac{1}{10M_1^2}\right) \tag{61}$$

Thus, the peak temperature attainable by microwave heat ing as given by Eq (59) is substantially lower than τ_i For $M_1 \gg 1$, $\tau_p \sim \frac{2}{3}\tau_i$

V Concluding Remarks

The problem of interaction between microwaves and the shock-produced, fully ionized gas, as considered here, represents an idealized case Because of its relative simplicity, some features of the physical problem can be understood by analytical methods Thus, the phenomenon of thermal choking, the moving shock wave solution, and the peak temperature of the gas have been analyzed In particular, the peak temperature is shown to be the upper bound of the mean gas temperature for a given initial Mach number Since the ionization of the gas is assumed to be completed, all of the microwave energy goes into heating of the gas Therefore, this peak temperature represents an upper bound for a given initial Mach number under all conditions

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